Spiral curves (Geometrical Models for Spatial Data Computations) (The 3-D Global Spatial Data Model)

A spiral is defined as a curve whose radius is inversely proportional to its length. Various kinds of spirals can be developed from the same definition. Mathematicians often view spirals from the perspective of an origin located at that point where the radius approaches zero. Spirals used in geomatics applications share the definition, but are viewed from a different perspective (i.e., beginning at that point where the spiral length is zero and the radius is infinitely long). This origin is selected because spirals are used to provide a rigorous gradual transition from traveling in a straight line along the tangent to traveling along a circular curve having a constant radius. Spirals are used extensively on railroad layout, but they are also used in some highway applications.

Typically, spirals are used in pairs. Starting on a straight-line tangent, a spiral is used to make the transition from traveling in a straight line to traversing a circular curve. At the end of the circular curve, another spiral is used to transition back to traveling in a straight line on the next tangent. Although the entrance spiral and exit spiral could have different lengths, standard practice is for them to be symmetrical. Therefore, the discussion here will focus only on the geometry of the entrance spiral.

Spiral Geometry

In some cases, spirals have been avoided because of their computational complexity. But, even though the equations have not changed, modern computers, coordinate geometry routines, and radial surveying techniques have eased the burden of using spirals. It is intended for the following to provide a comprehensive computational algorithm that can be completed using only a simple scientific calculator.

Admittedly, the formulas given here do not lend themselves to easy longhand solutions but will find ready applications in a spreadsheet solution or computer program. The equations presented here will likely be most useful to those writing coordinate geometry routines (for computers and/or data collectors).

Symbols for the spiral illustrated in Figure 4.11 are as follows:

α = azimuth of beginning tangent.
TS = station at transition tangent to spiral.
SC = station at transition spiral to circular curve.
L = total length of spiral (L also = SC-TS).
R = radius of circular curve at end of the spiral.
K = spiral constant = 1 / (2RL).
s = distance along spiral from TS to specified point on spiral.
r = instantaneous radius of spiral at location defined by s.
θ = angular difference between spiral radius vectors at two points. Typically one point is at the beginning of the spiral, and the second is any point on the spiral at a distance s from the beginning.
x = tangent component of the distance from TS to any point on the spiral.
y = perpendicular distance from the tangent to any point on the spiral. An appropriate sign convention is for y to be positive if the spiral curves to the right and negative if the spiral curves to the left when standing at TS and looking along the tangent adjacent to the spiral.

Equations for computing coordinates of a point on a spiral are given below.

Given:

e_1, n_1 = local plane coordinates of tangent to spiral, TS

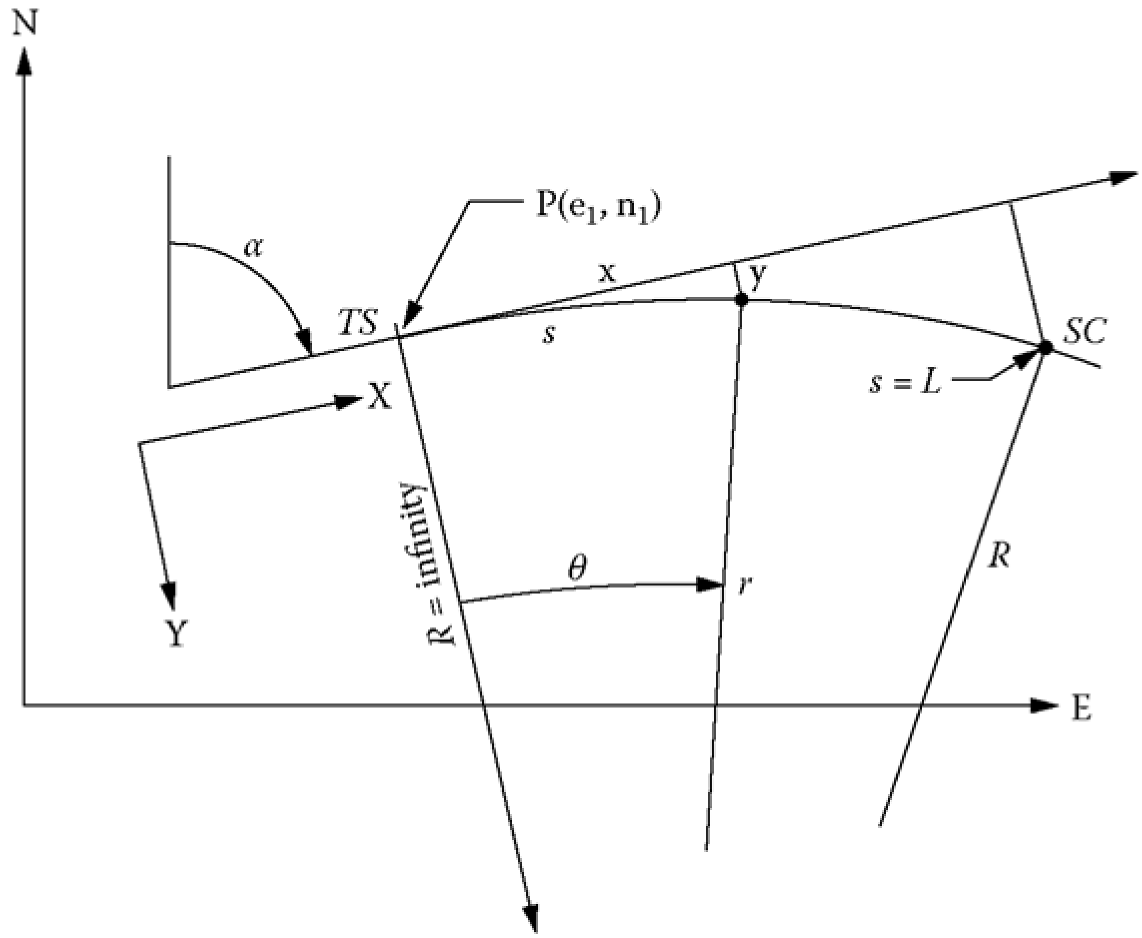

FIGURE 4.11 Spiral Elements and Geometry

α = local azimuth of straight-line tangent at *TS*
R = radius of circular curve at the end of the spiral
L = total length of the spiral
s = centerline distance from *TS* to point on the spiral

Store: constants common to spiral solutions (Libby and Booth, 1973, 8-9).

$$C_1 = 1/3$$
$$C_2 = -1/10$$
$$C_3 = -1/42$$
$$C_4 = 1/216$$
$$C_5 = 1/1,320$$
$$C_6 = -1/9,360$$
$$C_7 = -1/75,600$$
$$C_8 = 1/685,440$$

Compute:

$$\theta = s^2 / (2RL) = K\,s^2 \tag{4.38}$$

$$x = s\,(1 + C_2\,\theta^2 + C_4\,\theta^4 + C_6\,\theta^6 + C_8\,\theta^8 + \ldots)$$

$$= s\,(1 + \theta^2\,(C_2 + \theta^2\,(C_4 + \theta^2\,(C_6 + C_8\theta^2\,)))) \tag{4.39}$$

$$Y = s\,(C_1\,\theta + C_3\,\theta^3 + C_5\,\theta^5 + C_7\,\theta^7 + \ldots)$$

$$= s\;\theta\,(C_1 + \theta^2\,(C_3 + \theta^2\,(C_5 + C_7\,\theta^2\,))) \tag{4.40}$$

Solution: coordinates for a point on a spiral are computed using the COGO forward computation equations 4.8 and 4.9 as follows (sign convention: y is negative for spiral to left):

$$e_2 = e_1 + x \sin \alpha + y \sin (\alpha + 90^\circ)$$

$$= e_1 + x \sin \alpha + y \cos \alpha \qquad (4.41)$$

$$n_2 = n_1 + x \cos \alpha + y \cos (\alpha + 90^\circ)$$

$$= n_1 + x \cos \alpha - y \sin \alpha \qquad (4.42)$$

A line parallel to a spiral is not a spiral. But, points on an offset to a spiral, (e3, n3), can be computed using equations 4.43 and 4.44. To compute coordinates of points lying to the right of the spiral, use a plus offset distance. Points to the left of the spiral are computed using the offset distance as a negative value. The azimuth of the line (radius vector) perpendicular to the line tangent to the spiral at a point is (a + 90° + θ).

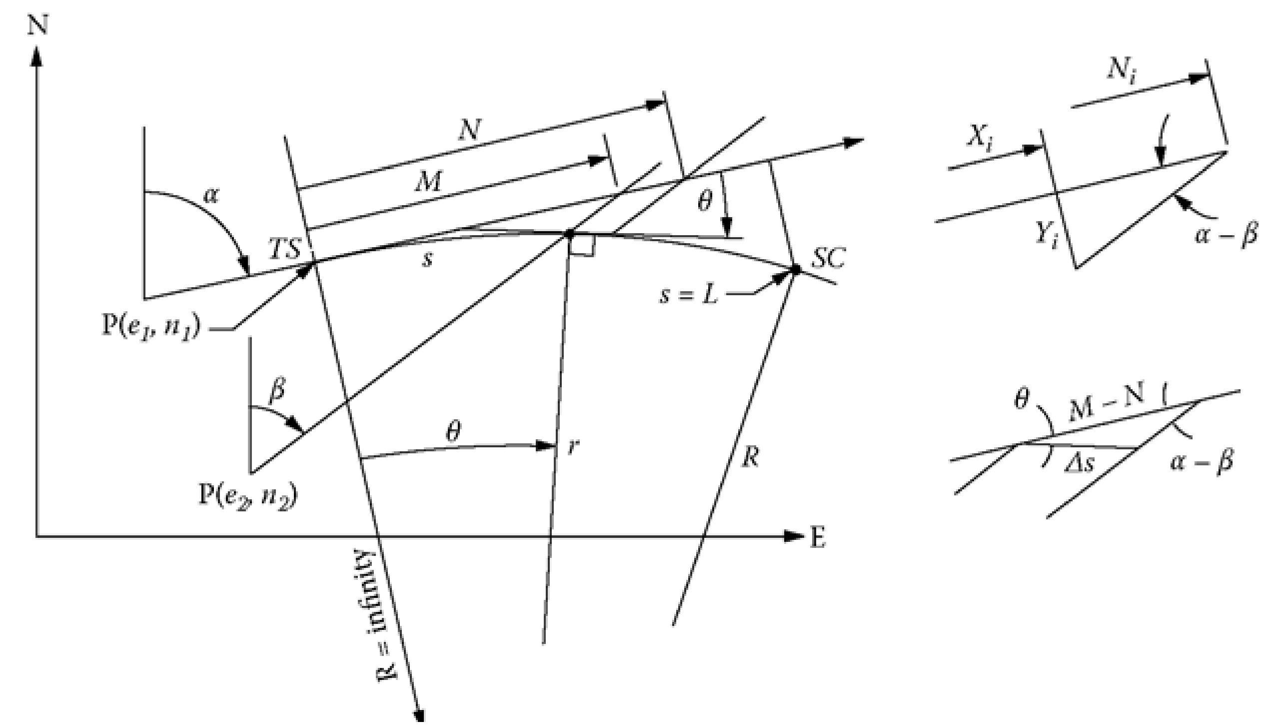

FIGURE 4.12 Spiral Intersection Elements and Geometry

$$e_3 = e_2 \pm \text{offset distance} * \sin (\alpha + 90^\circ + \theta) \qquad (4.43)$$

$$n_3 = n_2 \pm \text{offset distance} * \cos (\alpha + 90^\circ + \theta) \qquad (4.44)$$

With spiral points and offset line points computed according to equations 4.41, 4.42, 4.43, and 4.44, those points can be used along with other project points according to radial surveying techniques and standard coordinate geometry procedures available in most field computers and/or data collectors.

Intersecting a Line with a Spiral

Computing the intersection of a straight line with a spiral is not encountered very often, but there are times it needs to be done. Possible combinations include no intersection (trivial), one intersection (covered here), or, in extremely rare cases, two intersections. This section, as mentioned, looks at the single intersection case. As shown in Figure 4.12, the key to finding the solution is determining a correct value of s, the distance from the TS along the spiral centerline to the intersection. Once the value of s is known, the x and y spiral components are computed according to equations 4.39 and 4.40. Then the e and n coordinates of the intersection point are computed using equations 4.41 and 4.42. The value of s is determined using an iterative process.

Given:

e_1, n_1 = local plane coordinates of tangent to spiral, TS
α = local azimuth of straight-line tangent at TS
e_2, n_2 = local plane coordinates of any point on line
β = local azimuth from point on line to spiral intersection
R = radius of circular curve at the end of the spiral
L = total length of the spiral

Compute:

$K = 1 / (2RL)$.
M = line-line intersection distance along original tangent. Use equation 4.14 and distance d_1 as M. M does not change.
s_1 = value of M. The value of s will be improved.

Iterate: Start with $i = 1$, and continue incrementing i by 1 until tolerance is met.

θ_i = angle (in radians) to initial trial point on spiral = Ks_i^2.
x_i = trial distance along spiral tangent—see equation 4.30.
y_i = trial distance perpendicular to spiral tangent—see equation 4.31.
N_i = check distance; $N_i = x_i + y_i \cot(\alpha - \beta)$.
Tol = absolute value $|M - N|$. Is it small enough? If yes, quit.

If not:

$$\Delta s_i = \text{correction to } \Delta s \quad s = \frac{(M - N)\sin(\alpha - \beta)}{\sin(\alpha - \beta + \theta_i)}$$

$$s_{i+1} = s_i + \Delta s_i \text{ (}\Delta\ s_i \text{ should get smaller and smaller.)}$$

Increment i by 1.

Return to beginning of iteration using new value of i.

Solution: When the tolerance is met, s is the correct value.

The values of xi and have already been computed.

Use equations 4.41 and 4.42 to compute e and n at intersection.

As a final check, inverse from e2 and n2 to e and y. The computed direction should be the same as β.

Computing Area Adjacent to a Spiral

If a spiral is part of a boundary, computing the area adjacent to a spiral becomes an issue. The method presented here allows the user to compute the area between the original tangent and the spiral to any precision desired by choosing smaller and smaller increments of Δs and accumulating the area by numerical integration. Without a computer programmed to perform the repetitive calculations, the method loses its practicality. As shown in Figure 4.13, the area is accumulated from numerous trapezoids formed by the original tangent and perpendicular lines to the spiral. The separation of construction lines perpendicular to the tangent is not constant, but is determined by equal values of Δs on the spiral centerline.

Given:

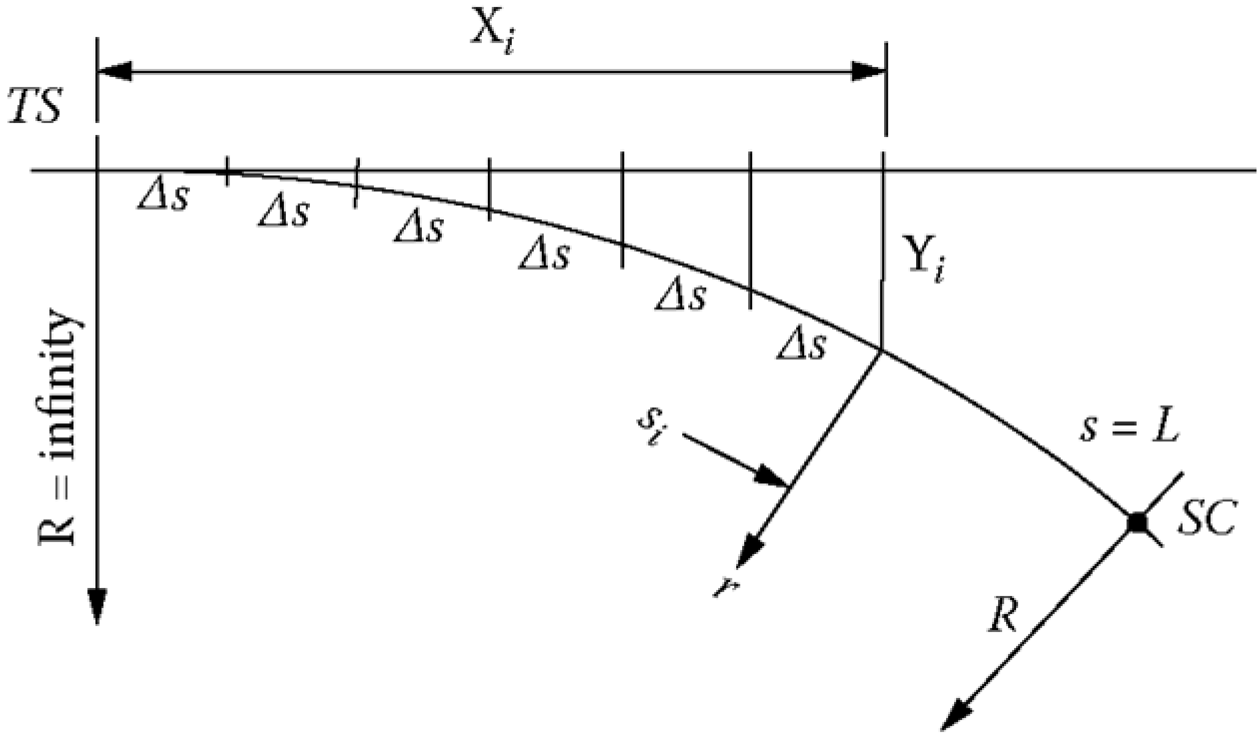

FIGURE 4.13 Area Adjacent to a Spiral

R = radius of circular curve at end of spiral

L = total length of spiral

s = L or some portion thereof

Find: area bounded by original tangent and spiral up to distance s.

Solution: K = 1/(2RL). Spiral constant.

Δs = increment chosen by user. $\Delta s = s/n$. User chooses n.
$s_0 = 0.0$.
$x_0 = 0.0$.
$y_0 = 0.0$.

Loop: for $i = 1$ to n:

s_i = distance to point i. $s_i = s_{i-1} + \Delta s$.
$\theta_i = Ks_i^2$. Angle in radians to point on spiral.
x_i = tangent component distance to point on spiral. See equation 4.39
y_i = perpendicular distance tangent to spiral. See equation 4.40.

End of loop.
Area by trapezoids:

$$2 \text{ area} = (x_1 - x_0)(y_1 + y_0) + (x_2 - x_1)(y_2 + y_1) \ldots$$
$$+ (x_n - x_{n-1})(y_n + y_{n-1})$$

which, after considerable algebraic manipulation, reduces to

$$2 \text{ area} = x_2y_1 - x_1y_2 + x_3y_2 - x_2y_3 \ldots. \quad + x_ny_{n-1} - x_{n-1}y_n + x_ny_n \tag{4.45}$$

Or, the area adjacent to a spiral can also be written as

$$2A = \sum_{i=2}^{n} x_iy_{i-1} - \sum_{i=2}^{n} x_{i-1}y_i + x_ny_n \tag{4.46}$$

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Previous post: [Circular Curves (Geometrical Models for Spatial Data Computations) (The 3-D Global Spatial Data Model)]

- **Related Links**
  - The 3-D Global Spatial Data Model
    - Introduction to the Global Spatial Data Model (GSDM) Defined